Math 206A Lecture 6 Notes

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1 Bárány's Theorem and Fractional Helly's Theorem

1.1 Proof of Bárány's theorem

We are now ready to prove Bárány's theorem.

Theorem 1.1 (Bárány)**.** *For every d, there exists a constant $\alpha_d > 0$ such that for every $X = \{x_1, \dots, x_n\} \subseteq \mathbb{R}^d$, there exists $z \in \mathbb{R}^d$ such that $z \in \mathrm{Con}(Z_I)$, $|I| = d+1$ for at least $\alpha_d \binom{n}{d+1}$ subsets I.*

Recall the two theorems we proved last time.

Theorem 1.2 (colorful Carathèodory)**.** *Let $X_1, \dots, X_{d+1} \subseteq \mathbb{R}^d$ be finite sets with $0 \in \mathrm{Conv}(X_i)$ for all i. Then there exist $x_1 \in X_1, x_2 \in X_2, \dots, x_{d+1} \in X_{d+1}$ such that $0 \in \mathrm{Conv}(\{x_1, \dots, x_{d+1}\})$.*

Theorem 1.3 (weak Tverberg)**.** *Let $r, d \in \mathbb{N}$. For every $n \geq (r-1)(d+1)^2 + 1$ and $x_1, \dots, x_n \in \mathbb{R}^d$, there exist $I_1, \dots, I_r \subseteq [n]$ with $I_i \cap I_j = \varnothing$ such that $\bigcap_{i=1}^r \mathrm{Conv}(X_{I_i}) \neq \varnothing$.*

We will show these two imply Bárány's theorem.

Proof. Choose $r = \lfloor n/(d+1)^2 \rfloor$. By weak Tverberg, there exist $X_1, \dots, X_r \subseteq X$ such that $\bigcap \mathrm{Conv}(X_i) \neq \varnothing$. Let $z \in \bigcap \mathrm{Conv}(X_i) \neq \varnothing$. By colorful Carathéodory, for all $(d+1)$-subsets of $[r]$, there exists a colorful simplex Δ which contains z. The number of such simplices is

$$\#\Delta = \binom{r}{d+1} = \binom{n/(d+1)^2}{d+1}.$$

Use the fact that $\binom{n}{k} > \frac{(n-k)!}{k!}$. Then

$$\#\Delta > \alpha_d \binom{n}{d+1}.$$

You can check that $\alpha_d \approx 1/d^d$. $\square$

1.2 Fractional Helly's theorem

Theorem 1.4 (fractional Helly)**.** *Fix $d, \alpha > 0$. Let $X_1, \dots, X_n \subseteq \mathbb{R}^d$ be convex sets such that at least $\alpha\binom{n}{d+1}$ of $(d+1)$-element sets $I \subseteq [n]$ have nonempty X_I. Then there exists $J \subseteq [n]$ such that $|J| > \alpha n/(d+1)$ and $X_J \neq \varnothing$.*

Lemma 1.1. *Without loss of generality, one can assume all X_i are convex polytopes.*

Proof. Replace each X_i with Y_i, where $Y_i = \mathrm{Conv}(\{y_I : i \in I\})$, where $y_I \in \bigcap_{i \in I} X_i$. This does not change any of the desired properties of the X_i. □

Definition 1.1. A **Morse function** $\varphi : \mathbb{R}^d \to \mathbb{R}$ is a linear function which is nonconstant on edges of the Y_i.

Lemma 1.2. *Let $I \subseteq [n]$, $Y_I \neq \varnothing$, and $v = \min_\varphi(Y_I)$. Then there exists $J \subseteq I$ such that $|J| \leq d$ and $v = \min_\varphi(Y_J)$.*

Proof. Apply the contrapositive of Helly's theorem where one of the subsets is the half space $H_- = \phi^{-1}((-\infty, \phi(v))$ and the other subsets are Y_i with $i \in I$. Then $\bigcap Y_i \cap H_- = \varnothing$, so the contrapositive of Helly's theorem gives $J \subseteq I$ such that $|J| \leq d$ and $\bigcap_{j \in J} Y_j \cap H_- = \varnothing$. □

We can now prove the theorem.

Proof. We have $\gamma : I \mapsto J$. Consider $I \subseteq [n]$ with $|I| = d+1$. From lemma 2, there exists some $J_0 \subseteq [n]$ with $|J_0| = d$ such that $J_0 = \gamma(I)$ for at least $\alpha\binom{n}{d+1}/\binom{n}{d} = \alpha\frac{n-d}{d+1}$ different I. Let $v = \min_\varphi(Y_{J_0})$. Thus, there exist at least $\alpha\frac{n-d}{d+1}$ $i \in I \setminus J_0$ such that $v \in Y_i$. So v is in at least $|J_0| + \alpha\frac{n-d}{d+1} = d + \alpha\frac{n-d}{d+1} > \alpha n/(d+1)$ convex subsets Y_i. □

Remark 1.1. The optimal bound is $1 - (1-\alpha)^{1/(d+1)}$ instead of $\alpha n/(d+1)$.